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Weakly bound states in $2+\epsilon$ dimensions

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Abstract. We study the critical behaviour near the threshold where a first bound state appears at some value of coupling constant in an attractive short-range potential in $2+\epsilon$ dimensions. We obtain a general expression for the binding energy near the threshold and also demonstrate that the critical region is correctly described by an effective separable potential. The critical exponent of the radius of weakly the bound state is shown to coincide with the correlation length exponent for the spin model in the large-N limit. In two dimensions, where the binding energy is exponentially small in coupling constant, we obtain a general analytic expression for the prefactor.

1. Introduction

It is well known, that in three dimensions a bound state for a particle in a short-range potential exists not at an arbitrary value of the coupling constant λ, but only at $\lambda \geqslant \lambda_c$, where λ_c is a critical value which depends on the particular potential. This may be viewed as the simplest example of a quantum phase transition, when e.g. an excitation gap vanishes as some parameter of the Hamiltonian is varied (see e.g. [1] for a review). In a sense, such a behaviour is similar to the second-order phase transition. Near the threshold the energy of the bound state behaves like an 'order parameter' $E \sim (\lambda - \lambda_c)^\beta$, where λ plays the role of temperature and β is the critical exponent. Deeper investigation of this critical behaviour is interesting in itself and may have some applications (see [2] and references therein).

The lower critical dimensionality for this transition is $d = 2$, since in two dimensions there always exists a bound state with energy exponentially small in λ [3], hence $\lambda_c = 0$ in this case. For this reason it seems natural to study the critical behaviour in $2+\epsilon$ dimensions, considering ϵ to be a small parameter, as was done for the phase transition in the nonlinear $O(N)$ sigma model [4] and also in the theory of Anderson localization (see e.g. [5]). It appears possible to develop an ϵ-expansion both for the wavefunction at the critical point and for the critical coupling $\lambda_c(\epsilon)$ [6, 7]. Even the first two terms of the expansion of $\lambda_c(\epsilon)$ in powers of ϵ give a rather accurate estimate for λ_c in three dimensions.

In this note we consider the onset of the first bound state in $2+\epsilon$ dimensions in more detail. First, we demonstrate that $\beta = 2/\epsilon$ for $\epsilon \leqslant 2$ and $\beta = 1$ above four dimensions. The result at small ϵ is not unexpected, since β must go to infinity as $\epsilon \to 0$ to reproduce the exponential dependence of E on λ in two dimensions. These expressions for β are consistent with the results of Lassaut *et al* [2]. Although they have studied the three-dimensional case with non-zero orbital momentum l, this is equivalent to the s-state problem in $2+\epsilon$ dimensions with $\epsilon = 1 + 2l$ (see below).

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However, the method used here is different from that of [2]. Starting from the integral representation of the Schrödinger equation we first derive exact results for the binding energy at $\lambda \sim \lambda_c$. Then we show also that the correct description of the critical region is given by the *separable* approximation, with the true interaction potential replaced by an effective nonlocal separable one, which depends on the zero-energy solution of the Schrödinger equation at $\lambda = \lambda_c$.

When $\lambda \to \lambda_c$ the radius of the bound state diverges as $(\lambda - \lambda_c)^{-\nu}$ with the critical exponent $\nu = \frac{1}{2}\beta$. Therefore the result obtained may be represented in terms of ν, namely $\nu = 1/\epsilon$ at $\epsilon \leqslant 2$ and $\nu = \frac{1}{2}$ at $\epsilon > 2$. This exponent coincides with the correlation-length exponent in the spherical model (equivalent to the N-component spin model at $N \to \infty$)[8] and with the localization-length exponent in the self-consistent theory of Anderson localization [9]. It is not clear whether this coincidence implies some nontrivial relation between these models, but still it seems rather interesting.

The same critical exponents were obtained from scaling considerations by Hwa and Nattermann [10] and Kolomeisky and Straley [11], who considered the problem of unbinding a directed polymer from a columnar defect in the presence of quenched disorder. Since such a polymer may be viewed as a worldline of a quantum particle, in the clean case this problem is essentially the same as the one discussed here.

Next, the approach used here also makes it possible to obtain an asymptotic expression for the energy of the bound state in two dimensions at $\lambda \to 0$ along with the pre-exponential factor. This general analytic expression for the prefactor seems, to the best of our knowledge, to be a new one. We also calculate this prefactor for some simple potentials and discuss its connection with the ϵ-expansion for λ_c.

As a direct application of the general result for the binding energy in two dimensions we also consider a case of a two-centre potential. The attractive force between the centres due to the bound state is shown to be of the Coulomb type (cf [12]) and the universal prefactor is found without solving the Schrödinger equation.

2. Critical behaviour in $2 + \epsilon$ dimensions

Consider the Schrödinger equation in d dimensions

$$-\Delta\Psi(\boldsymbol{r}) + \lambda V(\boldsymbol{r})\Psi(\boldsymbol{r}) = E\Psi(\boldsymbol{r}) \tag{1}$$

($\hbar = 1$, $2m = 1$), where λ is the coupling constant and $V(\boldsymbol{r})$ is a short-range attractive potential. We assume $V(\boldsymbol{r})$ to decrease faster then $1/r^2$ as $r \to \infty$ and shall deal here with the weakly bound state of the size $\sim (-E)^{-1/2} \gg a$, where a is the radius of the potential.

The first bound state in the problem appears at some critical value of the coupling constant $\lambda = \lambda_c$. Let us denote by ψ_0 the wavefunction of this state at the threshold. Then ψ_0 obeys the following zero energy equation

$$-\Delta\psi_0 + \lambda_c V(\boldsymbol{r})\psi_0 = 0. \tag{2}$$

It is convenient to normalize ψ_0 by the condition

$$\int \mathrm{d}\boldsymbol{r}\, \psi_0^2(\boldsymbol{r}) V(\boldsymbol{r}) = -1. \tag{3}$$

Note, that $\psi_0(\boldsymbol{r})$ need not be square integrable and the convergence of the normalization integral in (3) is guaranteed by the short-range potential $V(\boldsymbol{r})$.

We shall now determine both the critical value λ_c and the binding energy at $\lambda \sim \lambda_c$. For this purpose first rewrite the original Schrödinger equation in the integral form

$$\Psi(\boldsymbol{r}) = -\lambda \int \mathrm{d}\boldsymbol{r}'\, G_E(\boldsymbol{r}-\boldsymbol{r}')V(\boldsymbol{r}')\Psi(\boldsymbol{r}') \tag{4}$$

where $G_E(\boldsymbol{r})$ is the Green function of the free particle. This integral equation may also be viewed as an equation determining $\lambda(E)$ for the given negative energy E of the bound state. Multiplying both sides of (4) by $V(\boldsymbol{r})\Psi(\boldsymbol{r})$ and integrating over $\boldsymbol{r}$ we easily obtain

$$\frac{1}{\lambda} = -\frac{\int \mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}'\, V(\boldsymbol{r})\Psi(\boldsymbol{r})G_E(\boldsymbol{r}-\boldsymbol{r}')V(\boldsymbol{r}')\Psi(\boldsymbol{r}')}{\int \mathrm{d}\boldsymbol{r}\,\Psi^2(\boldsymbol{r})V(\boldsymbol{r})}. \tag{5}$$

If we put here $E = 0$ then $\Psi \to \psi_0$ and using the normalizing condition (3), we have

$$\frac{1}{\lambda_c} = \int \mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}'\, W(\boldsymbol{r})G_0(\boldsymbol{r}-\boldsymbol{r}')W(\boldsymbol{r}') \tag{6}$$

where

$$W(\boldsymbol{r}) = V(\boldsymbol{r})\psi_0(\boldsymbol{r}). \tag{7}$$

At zero energy $G_0(r)$ is merely the Green function of the Laplace operator and

$$G_0(r) = \frac{1}{\epsilon\sigma_\epsilon}\frac{1}{r^\epsilon} \tag{8}$$

(see e.g. [13]), where

$$\sigma_\epsilon = \frac{2\pi^{1+\epsilon/2}}{\Gamma(1+\epsilon/2)} \tag{9}$$

is the area of the unit sphere in $2+\epsilon$ dimensions. Then the critical value of the coupling constant may be represented as follows

$$\lambda_c = \epsilon\frac{2\pi^{1+\epsilon/2}}{\Gamma(1+\epsilon/2)}\left[\int \mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}'\,\frac{W(\boldsymbol{r})W(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^\epsilon}\right]^{-1}. \tag{10}$$

This expresion explicitly demonstrates that normally λ_c tends to zero as ϵ when we approach two dimensions.

Now, in the vicinity of the critical point we may write

$$\Psi = \psi_0 + \delta\Psi \qquad G_E = G_0 + \delta G_E \tag{11}$$

and assume all corrections to zero energy values to be small. If we substitute (11) in (5) and retain only terms of first order in $\delta\Psi$ we finally obtain

$$\frac{1}{\lambda} = \frac{1}{\lambda_c} + \int \mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}'\, W(\boldsymbol{r})\delta G_E(\boldsymbol{r}-\boldsymbol{r}')W(\boldsymbol{r}') \tag{12}$$

with $1/\lambda_c$ given by (6). Note that the terms containing $\delta\Psi$ cancel out. This cancellation is a consequence of the zero-energy equation

$$\psi_0(\boldsymbol{r}) = -\lambda_c \int \mathrm{d}\boldsymbol{r}'\, G_0(\boldsymbol{r}-\boldsymbol{r}')V(\boldsymbol{r}')\psi_0(\boldsymbol{r}') \tag{13}$$

and is actually due to the right-hand side of (5) being a variational functional which is stable against small variations of the true wavefunction [14]. In the limit $E \to 0$ we can expand $\delta G_E(\boldsymbol{r}-\boldsymbol{r}')$ in equation (12) in powers of E. In $2+\epsilon$ dimensions with $\epsilon < 2$ we have at small negative E

$$\delta G_E \simeq -\frac{\Gamma(1-\epsilon/2)}{2\epsilon\pi}\left(-\frac{E}{4\pi}\right)^{\epsilon/2} + \mathcal{O}(Er^{2-\epsilon}) \tag{14}$$

(see appendix A). Substituting this expression in (12), we finally obtain

$$E = -\left(A\frac{\lambda-\lambda_c}{\lambda}\right)^{2/\epsilon} \qquad 0<\epsilon<2 \tag{15}$$

where

$$A = 2^{\epsilon}\frac{\Gamma(1+\epsilon/2)}{\Gamma(1-\epsilon/2)}\frac{\int \mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}'\,W(\boldsymbol{r})W(\boldsymbol{r}')|\boldsymbol{r}-\boldsymbol{r}'|^{-\epsilon}}{(\int \mathrm{d}\boldsymbol{r}\,W(\boldsymbol{r}))^2}. \tag{16}$$

At $\epsilon > 2$, the leading term in the expansion (14) for $\delta G_E(r)$ is $\sim E$ and

$$E \sim (\lambda-\lambda_c) \qquad \epsilon \geqslant 2. \tag{17}$$

These equations formally solve the problem of the critical behaviour near the transition where the first bound state appears in $d = 2+\epsilon$ dimensions. In three dimensions equation (15) leads to $E \sim (\lambda-\lambda_c)^2$ and for the square-well potential one can easily verify (using ψ_0 from equation (49) of appendix B) that equations (10) and (16) give the correct answer $\lambda_c = \pi^2/4a^2$ and $A = \pi^2/8a$. Note, that if the Schrödinger equation is solved in the critical point, i.e. λ_c and ψ_0 are known, one can also evaluate the prefactor A.

The results obtained should be compared with that of [2], where the l-wave case was considered, because the radial s-wave Schrödinger equation in $d = 2+\epsilon$ dimensions is equivalent to the three-dimensional equation with nonzero orbital moment $l = (\epsilon-1)/2$ (see appendix A). Hence e.g. the dependence(15) is the same, as $E \sim (\lambda-\lambda_c)^{2/(2l+1)}$ obtained in [2] for $l < \frac{1}{2}$.

The particular form of equation (12) suggests that the correct description of the critical region near the threshold $\lambda \sim \lambda_c$ can be obtained within the separable approximation. This approximation, widely used in nuclear physics, involves replacing the original potential $V(\boldsymbol{r})$ with a nonlocal separable one, for which the Schrödinger equation is exactly solvable. In our case one should take

$$V_{\mathrm{sep}} = -V|\psi_0\rangle\langle\psi_0|V \tag{18}$$

where ψ_0 is the zero energy solution normalized by the condition (3). If the ground-state wavefunction $\Psi \approx \psi_0$ then V_{sep} is in a sense close to V, since $(V - V_{\mathrm{sep}})\psi_0 = 0$.

The Schrödinger equation for a particle in the potential (18) reads

$$-\Delta\Psi(\boldsymbol{r}) - \lambda V(\boldsymbol{r})\psi_0(\boldsymbol{r})\int \mathrm{d}\boldsymbol{r}'\,V(\boldsymbol{r}')\psi_0(\boldsymbol{r}')\Psi(\boldsymbol{r}') = E\Psi(\boldsymbol{r}) \tag{19}$$

and has an obvious solution for the bound state, which up to a normalizing constant is given by

$$\Psi(\boldsymbol{r}) = -\lambda\int \mathrm{d}\boldsymbol{r}'\,G_E(\boldsymbol{r}-\boldsymbol{r}')W(\boldsymbol{r}'). \tag{20}$$

The energy of the bound state is determined by substitution of (20) into equation (19), i.e. from the equation

$$1 = \lambda\int \mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}'\,W(\boldsymbol{r})G_E(\boldsymbol{r}-\boldsymbol{r}')W(\boldsymbol{r}'). \tag{21}$$

This is just the same equation as (12), since $G_E = G_0 + \delta G_E$ and $1/\lambda_c$ is determined from (6). Therefore the separable approximation (18) results in exact expressions (15), (17) in the close vicinity of the critical point. The validity of the separable approximation seems to be due to the wavefunction (20) having correct asymptotic behaviour at $r \gg a$. This is similar to the one-dimensional case, where the energy of the weakly bound state can be obtained by replacing the true $V(x)$ with a suitable δ-function potential, which also may

be viewed as a separable one. In appendix B we show how one can naturally arrive at the separable potential of the form (18).

As $\lambda \to \lambda_c$ the radius of the bound state $\xi \sim (-E)^{-1/2}$ goes to infinity as $(\lambda - \lambda_c)^{-\nu}$, where we have introduced a new critical exponent ν. Then from (15) and (17) it follows that

$$\nu = \begin{cases} \dfrac{1}{d-2} & 2 < d < 4 \\ \frac{1}{2} & d \geqslant 4. \end{cases} \tag{22}$$

Critical exponent ν diverges as $d \to 2$ and 'freezes' above $d = 4$ at the mean-field value $\nu = \frac{1}{2}$. This is precisely the correlation-length exponent for the N-component spin model at $N \to \infty$ [8]. Another model where equation (22) arises, is the self-consistent theory of Anderson localization for a particle in a random potential. In this case the localization length diverges as $(E_F - E_c)^{-\nu}$ if the Fermi energy E_F approaches the mobility edge E_c and ν is also given by equation (22) [9]. It was even argued that this result for ν is valid beyond the self-consistent approximation and might be an exact one [15, 16].

This interesting coincidence arises from the fact that in all these models resulting equation, determining the behaviour of the correlation length ξ, has the form similar to (21) with $E \to \xi^{-2}$. For example, in the self-consistent theory of Anderson localization the localization length ξ is given by the equation

$$1 = B E_F^{-2-\epsilon} \int_0^{q_0} \frac{\mathrm{d}q\, q^{1+\epsilon}}{q^2 + \xi^{-2}} \tag{23}$$

where B is some constant and q_0 is a momentum cut-off [9]. Comparing (23) with (21) we see that this is indeed the equation for the binding energy ξ^{-2} in an effective short-range separable potential with $\lambda \sim 1/E_F^{2+\epsilon}$, $a \sim 1/q_0$. If the Fermi energy increases, λ tends to zero and at $\lambda = \lambda_c$ the bound state disappears. This critical point obviously corresponds to the Anderson transition. Perhaps this is not a mere coincidence and some direct mapping between these models might be established.

To conclude this section we should like to mention that at small ϵ the critical exponents derived here can be obtained without actually solving the Schrödinger equation. This was done e.g. by Hwa and Nattermann [10] and Kolomeisky and Straley [11], who considered the problem of unbinding a directed polymer (i.e. the worldline of a quantum particle) from a columnar defect. In this case simple scaling arguments immediately lead to equation (22). In fact, one can take any quantity (not necessarily the free energy as in [10, 11]) depending on some scale and look at the perturbation theory in λ. Consider e.g. the Born series for the s-wave scattering amplitude $f(k)$ (see e.g. [17])

$$f(k) = f_1(k) + f_2(k) + \cdots \tag{24}$$

where at small k

$$f_1 \sim \lambda \tilde{V}(0) \qquad f_2 \sim \lambda^2 \int_0^\infty \mathrm{d}q\, q^{1+\epsilon} \frac{|\tilde{V}(q)|^2}{k^2 - q^2 + \mathrm{i}0} \tag{25}$$

and $\tilde{V}(q)$ is the Fourier transform of the potential. For the perturbation theory to be valid it is necessary that $f_2/f_1 \ll 1$. At small ϵ one has from (25) $f_2 \sim \lambda^2 \tilde{V}^2(0)(1 - (k/k_0)^\epsilon)/\epsilon$, where $k_0 \sim 1/a$, and hence the particle is essentially free on a scale k if

$$\frac{f_2}{f_1} \sim \frac{\lambda}{\lambda_c} \left[1 - \left(\frac{k}{k_0} \right)^\epsilon \right] \ll 1 \tag{26}$$

(cf [10]), where $\lambda_c \sim \epsilon\tilde{V}(0)$. If $\lambda > \lambda_c$ then (26) is not fulfilled at $k = 0$ and the particle is bound. However, for

$$k \gg \xi^{-1} \sim k_0 \left(\frac{\lambda - \lambda_c}{\lambda}\right)^{1/\epsilon} \tag{27}$$

one can neglect the potential. Hence ξ from (27) may be viewed as the radius of the bound state. The critical exponent obtained in this way is the same as (22).

3. Weakly bound states in two dimensions

In three dimensions neither λ_c nor A are known exactly for an arbitrary potential, since we can solve the zero-energy problem only in some special cases. However, the situation is different in two dimensions. In this case arbitrarily weak short-range attractive potential binds a particle, so that $\lambda_c = 0$. Then the solution to zero-energy equation (2) is obviously $\psi_0(r) =$ constant, which, according to the normalization condition (3), results in $\psi_0 = |\int \mathrm{d}\boldsymbol{r}\, V(\boldsymbol{r})|^{-1/2}$ and

$$W(\boldsymbol{r}) = \frac{V(\boldsymbol{r})}{|\int \mathrm{d}\boldsymbol{r}\, V(\boldsymbol{r})|^{1/2}}. \tag{28}$$

Next, in the limit $\epsilon \to 0$ we have $\lambda_c \sim \epsilon$ and $A \to 1$. Therefore in this limit the right-hand side of equation (15) for the binding energy turns to an exponential function. Then for $\kappa = \sqrt{-E}$ we have

$$\kappa = C \exp\left(-\frac{2\pi}{\lambda|\int \mathrm{d}\boldsymbol{r}\, V(\boldsymbol{r})|}\right) \tag{29}$$

where the prefactor C is determined from the expansion of A in powers of ϵ

$$A \simeq 1 + \epsilon \ln C + \cdots. \tag{30}$$

Expanding A from (16) in ϵ we obtain

$$C = \exp\left(\ln 2 - \gamma - \frac{\int \mathrm{d}\boldsymbol{r}\, \mathrm{d}\boldsymbol{r}'\, V(\boldsymbol{r})V(\boldsymbol{r}')\ln|\boldsymbol{r} - \boldsymbol{r}'|}{(\int \mathrm{d}\boldsymbol{r}\, V(\boldsymbol{r}))^2}\right) \tag{31}$$

where $\gamma = 0.577\ldots$ is Euler's constant. Thus, in two dimensions we have a general explicit expression for the energy of the weakly bound state. In contrast to the one-dimensional case, where at $a \to 0$ we may approximate any short-range potential by the δ-function and the binding energy depends only on one potential-dependent integral $\int \mathrm{d}x\, V(x)$, here there are two different integrals, one of which being nonlocal. In three dimensions no such general closed form for the energy is available even near the threshold $\lambda = \lambda_c$.

The double integral in (29) resembles the one encountered earlier in the ϵ-expansion for $\lambda_c(\epsilon)$. For the spherically symmetric potential at small ϵ one has [6]

$$\lambda_c(\epsilon) \simeq \lambda_1\epsilon + \lambda_2\epsilon^2 + \cdots \tag{32}$$

where

$$\lambda_1 = -\frac{1}{\int_0^\infty \mathrm{d}r\, rV(r)} \tag{33}$$

$$\lambda_2 = -\frac{1}{2}\frac{\int_0^\infty \mathrm{d}r\, r \int_0^\infty \mathrm{d}r'\, r'V(r)V(r')\ln\frac{r_>}{r_<}}{[\int_0^\infty \mathrm{d}r\, rV(r)]^3} \tag{34}$$

Table 1. Values for λ_1 and λ_2 from (32), $\bar{a}$ from (36) and the prefactor C from (31) for several different potentials $V(r)$.

$-V(r)$	$\lambda_1 a^2$	$\lambda_2 a^2$	$\bar{a}/a$	Ca
$\exp(-r/a)$	1	$\ln 2 - \frac{1}{4}$	$e^{1-\gamma}$	$e^{-3/4}$
$\theta(a-r)$	2	$\frac{1}{2}$	$e^{-1/2}$	$2e^{-\gamma+1/4}$
$(a/r)\exp(-r/a)$	1	$\ln 2$	$e^{-\gamma}$	1
$\exp(-r^2/a^2)$	2	$\ln 2$	$e^{-\gamma/2}$	$\sqrt{2}e^{-\gamma/2}$

($r_>$ ($r_<$) is the greater (lesser) of r, r'). Then, after some straightforward calculations, we obtain another expression for C

$$C = \frac{1}{\bar{a}} \exp\left(\ln 2 - \gamma - \frac{\lambda_2}{\lambda_1}\right) \tag{35}$$

where $\bar{a}$ is the mean range of the potential, defined by

$$\ln \bar{a} = \frac{\int_0^\infty \mathrm{d}r\, r \ln r\, V(r)}{\int_0^\infty \mathrm{d}r\, r\, V(r)}. \tag{36}$$

We evaluate λ_1, λ_2, $\bar{a}$ and C for several widely used potentials and the results are displayed in table 1. Note that the values of the prefactor C are surprisingly simple.

As mentioned earlier [6], equation (32) may be extrapolated to $\epsilon = 1$ to give a rather good estimate for the critical coupling in three dimensions. For the Yukawa potential this results e.g. in $\lambda_c a^2 \simeq 1 + \ln 2 \simeq 1.693\ldots$, which is close to the exact result $\lambda_c a^2 = 1.6798\ldots$.

It is also possible to use ϵ-expansion in the same manner to evaluate the prefactor A in three dimensions. From (30) it follows that $A^{1/\epsilon} \simeq C$ at small ϵ. Extrapolating this result to $\epsilon = 1$ we see that actually C may be treated as a first approximation for A in three dimensions. For the square-well potential this approximation gives $Aa \simeq 2e^{-\gamma+1/4} \simeq 1.442\ldots$ the exact value being equal to $Aa = \pi^2/8 = 1.234\ldots$. While qualitatively correct, this first approximation is not very accurate.

Equation (29) is valid for spherically nonsymmetric potentials as well, provided the range of the wavefunction κ^{-1} is much larger, than the radius a of a potential. Hence equation (29) may be used in the problem of several attractive centres.

In the case of two identical centres separated by a distance R we may write

$$V(\boldsymbol{r}) = v(\boldsymbol{r}) + v(\boldsymbol{r} + \boldsymbol{R}) \tag{37}$$

and upon substituting this potential in (31) we obtain that $C \sim \exp(-\frac{1}{2}\ln R)$ at $R \gg a$ for arbitrary short-range $v(\boldsymbol{r})$. Then for the energy E_2 of a weakly bound state at $a \ll R \ll \kappa^{-1}$ equations (29) and (31) yield

$$E_2 = -2e^{-\gamma}\frac{\kappa_0}{R} \tag{38}$$

where κ_0 is the square root of the binding energy on one centre (given by (29) and (31) with the replacement $V(\boldsymbol{r}) \to v(\boldsymbol{r})$). This is obviously the energy of the symmetric state. Note, that the energy E_2 depends on the details of the interaction only through κ_0.

It is interesting that the effective long-range force between two centres, resulting from the bound state is of the Coulomb type. In three dimensions the corresponding energy is known to behave as $1/R^2$ [18]. This is related to the collapse of three particle system with zero-range interaction, known as Thomas effect [19]. Less singular behaviour of the energy

(38) at small R is related to the absence of the Thomas effect in two dimensions [20]. For separable potential of a particular type the dependence κ_0/R in two dimensions was derived in [12]. We see now that this formula is quite universal.

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Appendix A

For the spherically symmetric potential the radial part of the $(2+\epsilon)$-dimensional Schrödinger equation reads

$$\left(-\frac{1}{r^{\epsilon+1}}\frac{\mathrm{d}}{\mathrm{d}r}r^{\epsilon+1}\frac{\mathrm{d}}{\mathrm{d}r}+\lambda V(r)\right)\Psi(r)=E\Psi(r). \tag{39}$$

The substitution $\Psi=\phi r^{-(1+\epsilon)/2}$ puts equation (39) in the form

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{(\epsilon^2-1)}{4r^2}-\lambda V(r)+E\right)\phi=0. \tag{40}$$

This is obviously the radial equation for the wavefunction with nonzero angular momentum l in three dimensions with $l(l+1)=(\epsilon^2-1)/4$, i.e. $l=(\epsilon-1)/2$.

Next we proceed to the evaluation of the free-particle Green function, $G_E(r)$, in $2+\epsilon$ dimensions. This function certainly can be found elsewhere in the literature but, for the sake of completeness we give here its short derivation. The Green function satisfies the equation

$$(-\Delta-E)G_E(\boldsymbol{r})=\delta(\boldsymbol{r}). \tag{41}$$

At $\boldsymbol{r}\neq 0$ we have for $f=r^{\epsilon/2}G_E(r)$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\epsilon^2}{4r^2}+E\right)f=0. \tag{42}$$

Then, at $E<0$, $f(r)$ (which goes to zero as $r\to\infty$) is proportional to the modified Bessel function $K_{\epsilon/2}(\kappa r)$, where $\kappa=\sqrt{-E}$, hence

$$G_E(r)\sim r^{-\epsilon/2}K_{\epsilon/2}(\kappa r). \tag{43}$$

At $E=0$ the Green function reduces to the fundamental solution of the Laplace equation, which in $2+\epsilon$ dimensions is given by equation (8). At $\kappa r\to 0$ and $\epsilon<2$ we have

$$\begin{aligned}K_{\epsilon/2}(\kappa r)&=\frac{\pi}{2}\frac{I_{-\epsilon/2}(\kappa r)-I_{\epsilon/2}(\kappa r)}{\sin(\epsilon\pi/2)}\\&=\frac{\pi}{2\sin(\epsilon\pi/2)}\left[\frac{(\kappa r/2)^{-\epsilon/2}}{\Gamma(1-\epsilon/2)}-\frac{(\kappa r/2)^{\epsilon/2}}{\Gamma(1+\epsilon/2)}+\mathcal{O}((\kappa r)^{2-\epsilon/2})\right].\end{aligned} \tag{44}$$

Using this asymptotics, identity $\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z)$ and comparing (43) with (8) we can fix unknown constant in (43)

$$G_E(r)=\frac{1}{(2\pi)^{1+\epsilon/2}}\left(\frac{\kappa}{r}\right)^{\epsilon/2}K_{\epsilon/2}(\kappa r). \tag{45}$$

Small κ behaviour of (45) gives rise to the expansion (14) for $\delta G_E=G_E-G_0$ in the main text.

Appendix B

To derive the separable potential (18) one can start from the following separable decomposition of a local potential $V(\boldsymbol{r})$

$$V(\boldsymbol{r}) = \sum_n \sigma_n V|\psi_n\rangle\langle\psi_n|V \tag{46}$$

where the set of functions $|\psi_n\rangle$ is determined from the eigenvalue equation

$$(\Delta + E)^{-1} V\psi_n = \eta_n(E)\psi_n \tag{47}$$

and $\sigma_n = \pm 1$ depending on the sign of $\langle\psi_n|V|\psi_n\rangle$ (see e.g. [21]). For negative E this equation is in fact the Schrödinger equation for the bound states, where ψ_n is the wavefunction of a bound state with energy E in the potential $1/\eta_n(E)V(\boldsymbol{r})$, i.e.

$$(-\Delta + 1/\eta_n V(\boldsymbol{r}))\psi_n = E\psi_n. \tag{48}$$

From this equation one can derive a normalizing condition, which in this case is known to be $\langle\psi_n|V(\boldsymbol{r})|\psi_m\rangle = \sigma_n\delta_{nm}$ [21].

Since we are interested in the weakly bound state with $E \simeq 0$, we may now take the limit $E \to 0$ in equation (48), i.e. we may define the set ψ_n with respect to the zero energy. In this case ψ_n's are the threshold wavefunctions and the corresponding values of $1/\eta_n(0)$ are critical values of coupling constant. The wavefunction ψ_0 of the first ground state at $E \to 0$ obeys the zero-energy equation (2), where $\lambda_c = 1/\eta_0(0)$ and η_0 is the largest eigenvalue in (47). We do not know ψ_n's exactly for an arbitrary potential, but e.g. for the attractive square well of radius a in three dimensions one can easily obtain for the s-states

$$\psi_n \sim \begin{cases} \dfrac{1}{r}\sin\sqrt{\lambda_n}r & r < a \\ (-1)^n\dfrac{1}{r} & r > a \end{cases} \qquad \lambda_n = 1/\eta_n = \left(\frac{1}{2}\pi + \pi n\right)^2 \frac{1}{a^2}. \tag{49}$$

In the critical region $\lambda \sim \lambda_c$ the wavefunction of the first bound state is very close to ψ_0, so it seems natural to retain only the term with ψ_0 in the expansion (46) as a zero approximation. This approach is similar to the pole approximation in the scattering theory, valid for the resonance scattering when there exist a weakly bound state.

Assume next that $\langle\psi_0|V|\psi_0\rangle \leqslant 0$ so that $\sigma_0 = -1$. Then, in the vicinity of the critical point, where the transition from zero to one bound state occurs, we arrive at the separable potential (18)

$$V \simeq V_{\text{sep}} = -V|\psi_0\rangle\langle\psi_0|V \tag{50}$$

where ψ_0 is normalized by the condition (3).

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